Spectrum shape functions in first-forbidden beta transitions

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The beta-decay differential decay rate is written in Behrens and Bühring [1] (called BB) as

$$d^5\Gamma = \frac{G^2}{(2\pi)^5} F_0 L_0 (W_0 - W)^2 p W dW d\Omega_e d\Omega_\nu \left(C(W) + D(W) \frac{p}{W} \hat{\mathbf{p}} \cdot \hat{\mathbf{k}} + \cdots \right) \tag{1}$$

where $\hat{\mathbf{p}}$ and $\hat{\mathbf{k}}$ are unit vectors in the directions of the electron and neutrino respectively. Here W_0 is the maximum value of the electron total energy W, p is the electron momentum, $p^2 = W^2 - 1$ in electron rest-mass units, and F_0L_0 is the Fermi function as defined by BB. The spectrum shape functions of interest here are denoted $C(W)$ and $D(W)$ and their ratio defines the beta-neutrino angular-correlation coefficient

$$a_{e\nu}(W) = \frac{D(W)}{C(W)}. \tag{2}$$

Starting from the BB formalism and invoking the impulse approximation, we have derived simple expressions for the spectrum shape functions in terms of six nuclear matrix elements characterizing the parity-changing nature of first-forbidden beta decay. To display these expressions, we define

$$C(W) = \sum_K k_1^{(K)} \left(1 + A_1^{(K)} W + \frac{B_1^{(K)}}{W} + C_1^{(K)} W^2 \right)$$

$$D(W) = \sum_K k_2^{(K)} \left(1 + A_2^{(K)} W + \frac{B_2^{(K)}}{W} + C_2^{(K)} W^2 \right) \tag{3}$$

Where K is the multipolarity of the beta transition, which for first-forbidden decays has the value $K = 0$ and 1 for non-unique transitions and $K = 2$ for unique transitions. Expressions for the coefficients $k^{(K)}$, $k^{(K)} A^{(K)}$, $k^{(K)} B^{(K)}$ and $k^{(K)} C^{(K)}$ are given in Table I. They depend on the six nuclear matrix elements, on W_0 the maximum electron energy, and on $\xi = \alpha Z/2R$, where R is the radius of a uniformly charged sphere approximating the nuclear charge density distribution. The six nuclear matrix elements are defined schematically by

$$x = -g_V \langle ir C_1 \rangle \tag{4}$$

$$\begin{Bmatrix} w \\ u \\ z \end{Bmatrix} = g_A \begin{Bmatrix} \sqrt{3} \\ \sqrt{2} \\ -2 \end{Bmatrix} \langle ir [C_1 \times \sigma]^{(K)} \rangle \tag{5}$$

$$\xi' v = -\frac{g_A}{M}\sqrt{3}\langle[\mathbf{p}\times\sigma]^{(0)}\rangle \tag{6}$$

$$\xi' y = -\frac{g_V}{M}\langle\mathbf{p}\rangle \tag{7}$$

Table I. Expressions for the parameters introduced in the spectral function, Eq. (3), in terms of the nuclear matrix elements.

		$K=0$	$K=1$	$K=2$
$C(W)$	$k_1^{(K)}$	$\zeta_0^2+\frac{1}{9}w^2$	$\zeta_1^2+\frac{1}{9}(u+x)^2-\frac{2}{9}u(u+x)$ $+\frac{1}{18}W_0^2(2x+u)^2-\frac{1}{18}(2x-u)^2$	$\frac{1}{12}(W_0^2-1)z^2$
	$k_1^{(K)}A_1^{(K)}$	0	$-\frac{4}{3}\zeta_1 u-\frac{1}{9}W_0(2x+u)^2$	$-\frac{1}{6}W_0 z^2$
	$k_1^{(K)}B_1^{(K)}$	$-\frac{2}{3}\zeta_0 w$	$\frac{2}{3}\zeta_1(u+x)$	0
	$k_1^{(K)}C_1^{(K)}$	0	$\frac{4}{9}u^2+\frac{1}{18}(2x+u)^2+\frac{1}{18}(2x-u)^2$	$\frac{1}{6}z^2$
$D(W)$	$k_2^{(K)}$	$\zeta_0^2-\frac{1}{9}w^2$	$-\frac{1}{3}\zeta_1^2+\frac{1}{27}(u+x)^2-\frac{4}{9}W_0\zeta_1(2x+u)$ $-\frac{4}{27}(u+x)(2x-u)+\frac{1}{54}W_0^2(2x+u)^2$ $-\frac{1}{54}(2x-u)^2$	$-\frac{1}{60}(W_0^2-1)z^2$
	$k_2^{(K)}A_2^{(K)}$	0	$\frac{4}{3}\zeta_1 u+\frac{8}{27}W_0 u(2x+u)$ $+\frac{1}{27}W_0(4x^2-u^2)-\frac{1}{27}(2x+u)^2$	$\frac{1}{5}W_0 z^2$
	$k_2^{(K)}B_2^{(K)}$	0	0	0
	$k_2^{(K)}C_2^{(K)}$	0	$-\frac{20}{27}u^2+\frac{8}{27}x^2$	$-\frac{1}{5}z^2$

Here the spherical harmonic of rank L is written $C_L(\hat{\mathbf{r}}) = (4\pi/(2L+1))^{1/2}\, Y_L(\hat{\mathbf{r}})$. Further, $\mathbf{p}$ is the momentum operator, which in coordinate space is taken as a symmetrized derivative operator $-i\frac{1}{2}\left(\overleftarrow{\nabla}+\overrightarrow{\nabla}\right)$. Lastly, g_V and g_A are the vector and axial-vector coupling constants of the weak interaction, and M is the nucleon mass in electron rest-mass units. We have also introduced ζ_0 and ζ_1 as specific combinations of nuclear matrix elements:

$$\zeta_0 = \xi' v + \xi\omega' + \frac{1}{3}\omega W_0$$
$$\zeta_1 = \xi' y - \xi(u'+x') + \frac{1}{3}(u-x)W_0 \tag{8}$$

Most first-forbidden non-unique decays studied have been low-energy transitions in heavy nuclei for which the inequality $\xi \gg W_0$ holds. In this limit the coefficient $k^{(K)}$ is of order ξ^2, coefficients $k^{(K)}A^{(K)}$ and $k^{(K)}B^{(K)}$ are of order ξ and coefficient $k^{(K)}C^{(K)}$ is of order one. Thus in the ξ-approximation (in which only terms in the leading power of ξ are retained), the spectral functions $C(W)$ and $D(W)$ both become energy independent, and the beta spectrum has the allowed shape. The effect is independent of the nuclear matrix elements and thus little nuclear-structure information may be

determined from shape measurements alone. The beta-neutrino correlation coefficient in the ξ-approximation becomes energy independent and takes a simple form

$$a_{ev} = \frac{\zeta_0^2 - \frac{1}{3}\zeta_1^2}{\zeta_0^2 + \zeta_1^2} \tag{9}$$

Thus in a $0^+ \rightarrow 0^-$ transition in which only the rank-0 multipole enters, $a_{ev} = +1$, while in a $0^+ \rightarrow 1^-$ transition in which only rank-1 multipole enters, $a_{ev} = -1$. This is exactly the same situation that exists in allowed decays: $a_{ev} = +1$ for pure Fermi transitions and $a_{ev} = -1$ for pure Gamow-Teller transitions. Only for transitions for which both multipolarity 0 and 1 contribute is there any likelihood there will be any significant dependence on the details of nuclear matrix elements.

To test our coding, we have recomputed shell-model estimates of first-forbidden β-decay matrix elements for a few cases in light nuclei that are available in the literature. We start with a 1972 calculation of Towner and Hardy [2] for the $\frac{1}{2}^+ \leftrightarrows \frac{1}{2}^-$ transitions in ^{15}C, ^{17}N and ^{17}Ne. We compute the integrated spectrum shape function, *f*, defined as

$$f = \int_1^{W_0} Wp(W_0 - W)^2 F_0 L_0 C(W) dW. \tag{10}$$

Since this latter quantity includes all the matrix-element information, first forbidden *f t* values (up to radiative and isospin-symmetry-breaking corrections) are all equal to a universal constant established from the average corrected $\mathcal{Ft}$ value from superallowed beta decay: i.e.

$$ft = 2\mathcal{Ft} = 6145 \text{ s}. \tag{11}$$

Thus an experimental value for *f* can be deduced from the measured partial half-life via Eq. (11). The results are listed in Table II. Our calculated matrix elements and the integrated spectrum shape function are in very good agreement with the published values. The experimental values for *f* obtained from the ENSDF [3] listings of partial half-lives are in excellent agreement with the shell-model results.

The other literature values we compare with are those of Millener *et al.* [4] for the β decay of ^{11}Be ($\frac{1}{2}^+$) to the ground state ($\frac{3}{2}^-$)and first excited state ($\frac{1}{2}^-$) in ^{11}B. Millener's shell-model calculation is based on six valence orbitals:$1s_{1/2}$, $p_{3/2}$, $p_{1/2}$, $d_{5/2}$, $2s_{1/2}$, and $d_{3/2}$ using the Cohen-Kurath [5] effective interaction for *p*-shell interactions, and the Millener-Kurath [6] effective interaction for the cross-shell interactions. No more than one nucleon occupies the *sd*-shell orbitals in the $1 \rightarrow 0\hbar\omega$ calculation, and no more than two in the $1 \rightarrow (0 + 2)\hbar\omega$ calculation. The inclusion of the lowest $1s_{1/2}$ orbital is essential for the $2\ \hbar\omega$ calculation, in order that the spurious centre-of-mass motion could be correctly removed. In our

work, we only use five orbitals, $p_{3/2}$, $p_{1/2}$, $d_{5/2}$, $2s_{1/2}$ and $d_{3/2}$, and do not remove the spurious centre-of-mass component.

Table II. Nuclear matrix elements (in fm units) for $\frac{1}{2}^+ \leftrightarrows \frac{1}{2}^-$ transitions in the β decay of ^{15}C, ^{17}N and ^{17}Ne from the published values of Towner and Hardy [2] and the present work. Also given is the integrated spectrum shape function f and its current experimental value.

	$^{15}C(\beta^-)^{15}N$		$^{17}N(\beta^-)^{17}O$		$^{17}Ne(\beta^+)^{17}F$	
	TH72	this work	TH72	this work	TH72	this work
w	−1.854	−1.882	0.734	0.777	−0.734	−0.777
$\xi' v$	46.454	47.051	−16.025	−16.993	14.596	15.422
x	0.927	0.930	0.425	0.438	0.425	0.438
u	2.085	2.099	0.850	0.904	−0.850	−0.904
$\xi' y$	23.130	23.238	9.190	9.582	8.341	8.696
f	841	873	34	39	402	441
Expt.[3]	$f = 935 \pm 20$		$f = 45 \pm 7$		$f = 350 \pm 60$	

Our results are shown in Table III. For the small space $1 \rightarrow 0\ \hbar\omega$ calculation, our values of the coordinate matrix elements, *w, x, u, z*, agree quite well with Millener's values, but we are less successful in the $1 \rightarrow (0 + 2)\ \hbar\omega$ calculation. This is probably due to our failure to remove the spurious centre-of-mass component from the $2\hbar\omega$ wave functions. For the relativistic matrix elements, we have used Siegert's theorem for both $\xi\, v$ and $\xi\, y$. Millener uses a similar but not identical CVC relation for $\xi\, y$ and derivatives of oscillator functions for ξv. So our results differ from Millener's on the value of the

Table III. Nuclear matrix elements (in fm units) for the β decay of ^{11}Be ($\frac{1}{2}^+$) to the ground state ($\frac{3}{2}^-$) and first excited state ($\frac{1}{2}^-$) in ^{11}B from the published values of Millener *et al.*[4] and the present work in a small shell-model space, $1\rightarrow 0\ \hbar\omega$ and a large space $1 \rightarrow (0+2)\ \hbar\omega$. Also given is the integrated spectrum shape function for each multipole, $f^{(K)}$, and their sum: $f = \sum_K f^{(K)}$.

Final state		Millener *et al.* $1 \rightarrow 0\ \hbar\omega$	present work $1 \rightarrow 0\ \hbar\omega$	Millener *et al.* $1 \rightarrow (0+2)\ \hbar\omega$	present work $1 \rightarrow (0+2)\ \hbar\omega$
$\frac{1}{2}^-$	w	−1.206	−1.243	−1.178	−1.023
	$\xi' v$	35.840	27.417	30.160	22.569
	x	0.456	0.462	0.453	0.313
	u	1.387	1.404	1.350	0.969
	$\xi' y$	9.305	10.184	9.244	6.907
	$f^{(0)}$	465	206	293	140
	$f^{(1)}$	38	42	37	20
	f	503	248	330	160
	Expt. [4]	$f^{(0)} = 115 \pm 18$	$f^{(1)} < 35$	$f = 140 \pm 8$	
$\frac{3}{2}^-$	x	0.532	0.535	0.544	0.405
	u	−1.054	−1.027	−0.971	−0.755
	$\xi' y$	13.080	14.015	13.360	10.624
	z	1.816	1.977	1.900	1.520
	f	415	476	425	274
	Expt. [4]	$f = 244 \pm 9$			

relativistic matrix elements and hence on the integrated spectrum shape, *f*. Both calculations in the small space, $1 \rightarrow 0\ \hbar\omega$, find *f* values much larger than the experimental value. In the larger space, $1 \rightarrow (0 + 2)\ \hbar\omega$, our results are closer to experiment than Millener's because our relativistic matrix elements are smaller. This result gives some support for the use of Siegert's theorem in the evaluation of relativistic matrix elements.

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